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Uniting Gross–Neveu and massive Schwinger models

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Abstract

We show that it is possible to obtain the Gross–Neveu model in $1 + 1$ dimensions from gauge fields only. This is reminiscent of the fact that in $1 + 1$ dimensions the gauge field tensor is essentially a pseudo-scalar. We also show that it is possible in this context to combine the Gross–Neveu model with the massive Schwinger model in the limit where the fermion mass is larger than the electric charge.

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1. Introduction

Exactly solvable models in $1 + 1$ dimensions form a good theoretical laboratory due the possibility that they might provide ways to study and understand aspects of their comparably more complicated $(3 + 1)$ -dimensional cousins. In this Letter we are concerned about two of these; the (massive) Schwinger (SWM) [1] and the Gross–Neveu (GNM) [2] models. The former provides ways to understand the anomaly-gauge invariance relation, confinement, screening and so on, where as the latter presents spontaneous breaking of a discrete symmetry and all the interesting phenomena that is associated with it such as dynamical mass generation. Both these theories escape the Coleman [3] theorem in their own way: SWM in that the anomaly already explicitly breaks the continuous chiral symmetry of the fermion field and the GNM in that the symmetry that is spontaneously broken is discrete.

The common technique for solving the GNM is to introduce an auxiliary scalar field with a Yukawa type coupling to the fermion, so that using its equations of motion will yield the quartic fermion term of GNM. On the other hand in $1 + 1$ dimensions the electromagnetic tensor is essentially a pseudo scalar and there is a possibility to add a non-minimal coupling between fermions, the 2D counterpart of the $g - 2$ coupling in 4D, being essentially a term of Yukawa type. In this Letter we pursue this idea and show that it is indeed possible to get GNM from gauge fields and finally combine it with the massive SWM.

2. Gross–Neveu from gauge fields

In complete analogy to the use of scalars for studying the GNM we consider the following Lagrangian with N fermion and one vector fields,

$$\mathcal{L}_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\not{\partial}\Psi + \frac{g}{2}\epsilon_{\mu\nu}F^{\mu\nu}\bar{\Psi}\Psi. \quad (1)$$

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As it stands this Lagrangian is not renormalizable if one insists on using the vector field as the dynamical variable. The reason is that the 1-loop correction to the A_μ propagator will be proportional to $g^2 N \epsilon^{\mu a} \epsilon^{\nu b} q_a q_b$ and there are no counter-terms in (1) to compensate for the infinity that arises. Thus we modify (with the inclusion of a gauge fixing term) the Lagrangian as follows:

$$\mathcal{L}_2 = \mathcal{L}_1 + \frac{A}{8} (\epsilon_{\mu\nu} F^{\mu\nu})^2 - \frac{1}{2\xi} (\partial_\mu A^\mu)^2. \quad (2)$$

We argue as follows for the omission at this stage of the minimal coupling, $e A_\mu \bar{\Psi} \gamma^\mu \Psi$. The Lagrangian (2) is invariant under the discrete symmetry S defined to be

$$S \equiv P \times \{\Psi \rightarrow \gamma^5 \Psi \text{ and } A_\mu \rightarrow A_\mu\}, \quad (3)$$

where P stands for the usual parity transformation. Now if one includes the minimal coupling this symmetry will be explicitly broken by the $U(1)$ anomaly. However, as we will show shortly the Lagrangian (3) will exhibit spontaneous breaking (SSB) of the S symmetry. Thus ignoring the explicit breaking term is plausible if its scale is much smaller than the scale of the SSB, which, in the present case can be identified with the fermion mass M_f that would arise from SSB. So we have from the outset $e \ll \sim M_f$ and we can turn on the minimal coupling after the SSB occurs. More on this in Sections 3 and 4.

Varying the action with respect to A_μ (and ignoring for now the gauge fixing term) we get

$$\partial_\mu \left[-F^{\mu\nu} + \frac{A}{2} \epsilon^{\mu\nu} (\epsilon_{ab} F^{ab}) + g \epsilon^{\mu\nu} \bar{\Psi} \Psi \right] = 0. \quad (4)$$

If we replace $F_{\mu\nu} = \epsilon_{\mu\nu} \sigma$ we get $\sigma = g \bar{\Psi} \Psi / (1 + A)$ which gives the following Lagrangian

$$\mathcal{L}_{\text{GN}} = i \bar{\Psi} \not{\partial} \Psi - \frac{g^2}{2(1+A)} (\bar{\Psi} \Psi)^2, \quad (5)$$

which means that to have a correspondence to the GNM at all we have to demand

$$G^2 \equiv -\frac{g^2}{1+A} > 0, \quad (6)$$

that is:

$$1 + A < 0. \quad (7)$$

To proceed further we need the free photon propagator which is given by the terms of the Lagrangian

quadratic in A_μ ,

$$P^{\mu\nu} = \frac{-i}{q^2} \left[g^{\mu\nu} - \left(1 - \frac{1}{\xi} \right) \frac{q^\mu q^\nu}{q^2} + \left(\frac{A}{1+A} \right) \frac{\bar{q}^\mu \bar{q}^\nu}{q^2} \right]. \quad (8)$$

Here we have defined

$$\bar{q}^\mu \equiv \epsilon^{\mu\nu} q_\nu. \quad (9)$$

Now, as in the GNM, we invoke the large N argument. That is we let $N \rightarrow \infty$ keeping $g^2 N$ finite. Then the only $O(N^{-0})$ contributions come from the fermion loops. Since there will be an ultraviolet infinity that will arise from a fermion loop and since it will have the tensor form $\bar{q}^\mu \bar{q}^\nu$ the infinity will be gotten rid of by renormalizing A . A simple calculation using dimensional analysis and $\overline{\text{MS}}$ subtraction scheme yields the following

$$P^{\mu\nu} = \frac{-i}{q^2} \left[g^{\mu\nu} - \left(1 - \frac{1}{\xi} \right) \frac{q^\mu q^\nu}{q^2} + \left(\frac{B}{1+B} \right) \frac{\bar{q}^\mu \bar{q}^\nu}{q^q} \right], \quad (10)$$

with

$$1 + B = 1 + A(\mu^2) - \frac{g^2 N}{2\pi} \ln \left(\frac{-q^2}{\mu^2} \right), \quad (11)$$

$$\mu \frac{\partial A}{\partial \mu} \equiv \beta(A) = -\frac{g^2 N}{\pi}. \quad (12)$$

From the last equation it is easy to see that the theory is asymptotically free:

$$\mu \frac{\partial G^2}{\partial \mu} = -\frac{g^2 N}{\pi} \frac{g^2}{(1+A)^2}. \quad (13)$$

Clearly the propagator (10) has tachyonic poles and following the common wisdom there should be two meanings for this. Either the theory does not make sense at all or we are simply expanding about the wrong vacuum. We will now show that the latter is the case. To study a possible SSB it is enough to consider $F_{\mu\nu} = \epsilon_{\mu\nu} \sigma$ and study the theory around a constant σ_{cl} background field. The procedure is exactly the same as in the GNM [2,4]. Thus the renormalized effective potential is

$$V = -\frac{1+A}{2} \sigma_{\text{cl}}^2 + \frac{g^2 N}{4\pi} \sigma_{\text{cl}}^2 \left[\ln \left(\frac{\sigma_{\text{cl}}^2}{\sigma_0^2} \right) - 3 \right]. \quad (14)$$

Where we have introduced the non-zero subtraction field strength σ_0 (also an $\overline{\text{MS}}$ quantity) in such a way that $V''(\sigma_0) = -(1+A) > 0$. Since to the order we are working at there is no wave function renormalization the effective potential obeys the following renormalization group equation

$$\left[\sigma_0 \frac{\partial}{\partial \sigma_0} + \tilde{\beta}(A) \frac{\partial}{\partial A} \right] V = 0. \quad (15)$$

From here we find $\tilde{\beta}(A) = \beta(A)$ meaning $\mu = \text{const} \times \sigma_0$.

The potential (14) has two minima that are images of each other with respect to the symmetry S :

$$\sigma_{M_f} = \pm \sigma_0 \exp[1 + (1+A)\pi/g^2 N], \quad (16)$$

$$V''(\sigma_{M_f}) = \frac{g^2 N}{\pi}. \quad (17)$$

This signals the spontaneous breaking of S and consequently the fermion acquires a mass

$$M_f = g|\sigma_{M_f}|. \quad (18)$$

This mass being a physical quantity obeys the same renormalization group equation as the effective potential.

We can now compute the photon propagator in the broken symmetry phase of the theory. The renormalization condition we have employed for the effective potential means that

$$[\bar{q}_\mu \bar{q}_\nu \Pi^{\mu\nu}(q^2, M_f^2)]_{q^2=0}^{-1} = i V''(\sigma_{M_f}) = \frac{g^2 N}{\pi}. \quad (19)$$

Thus we have to calculate the loop with a non-zero fermion mass and we subtract at zero momentum. The above condition yields $\mu = g\sigma_0$ and we get the propagator in the broken phase to be

$$\Pi^{\mu\nu} = \frac{-i}{q^2} \left[g^{\mu\nu} + \left(\frac{1+C}{C} \right) \frac{\bar{q}^\mu \bar{q}^\nu}{q^2} \right], \quad (20)$$

$$C = -\frac{g^2 N}{2\pi} f(q^2/4M_f^2), \quad (21)$$

$$f(x) = 2\sqrt{\frac{1-x}{x}} \tan^{-1} \left[\sqrt{\frac{x}{1-x}} \right]. \quad (22)$$

We see that now there is a physical pole that appears at the threshold $4M_f^2$ and the tachyon has disappeared. This completes the full correspondence to the GNM because the 4-fermi amplitudes will come

to be exactly the same. For example the $e^+e^- \rightarrow e^+e^-$ scattering amplitude will be

$$\frac{2\pi i}{N} \left[\frac{1}{f(s/4M_f^2)} + \frac{1}{f(u/4M_f^2)} \right]. \quad (23)$$

3. Turning on the minimal coupling

It is better to study the theory with the minimal coupling turned on (that is now the fermion has charge) in the unbroken phase to get a better feeling about the SSB. When $e \neq 0$ there is another diagram that contributes to the photon propagator, but now with a different tensor structure:

$$i \frac{e^2 N}{\pi} \left[g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right]. \quad (24)$$

Here we again resort to large N argument. That is we let $N \rightarrow \infty$ keeping $e^2 N$ finite. As we did before summing all the 1PI graphs, the exact propagator becomes,

$$\Pi^{\mu\nu} = \frac{-i}{(q^2 - \frac{e^2 N}{\pi})} \left[g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} + \left(\frac{B}{1+B - e^2 N/(\pi q^2)} \right) \frac{\bar{q}^\mu \bar{q}^\nu}{q^2} \right]. \quad (25)$$

There will be tachyonic poles in this propagator if

$$1 + A(\mu^2) - \frac{g^2 N}{2\pi} \ln \left(\frac{-q^2}{\mu^2} \right) - \frac{e^2 N}{q^2 \pi} = 0. \quad (26)$$

It can be shown that the tachyons exist for all values of e and g . So we see that ignoring the minimal term before SSB is verified since this interaction term does not make the situation any better.

We now turn back to the broken phase. In this situation the contribution from the minimal coupling term changes due to the finite fermion mass. The full interacting propagator is given by:

$$\Pi^{\mu\nu} = -i X(q^2) \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) - i Y(q^2) \frac{\bar{q}^\mu \bar{q}^\nu}{q^2} - i \xi \frac{q^\mu q^\nu}{q^4}, \quad (27)$$

with

$$X(q^2) = \frac{1}{q^2(1 - \kappa F(z))}, \quad (28)$$

$$Y(q^2) = \frac{1}{q^2(1 - \kappa F(z))} \frac{1 + \lambda G(z)}{\lambda G(z) + \kappa F(z)}. \quad (29)$$

The functions F and G are:

$$G(z) = \sqrt{\frac{1-z}{z}} \tan^{-1} \left(\sqrt{\frac{z}{1-z}} \right), \quad (30)$$

$$F(z) = \frac{1}{z} \left[1 - \frac{1}{\sqrt{z(1-z)}} \tan^{-1} \left(\sqrt{\frac{z}{1-z}} \right) \right], \quad (31)$$

and we adopted the following for convenience in notation

$$z \equiv \frac{q^2}{4M_f^2}, \quad \lambda \equiv \frac{g^2 N}{\pi}, \quad \kappa \equiv \frac{e^2 N}{4\pi M_f^2}. \quad (32)$$

The functions X and Y have poles at $q^2 = 0$ and $q^2 = 4M_f^2 - (e^2 N)/(g^2 N)$ for $e \ll 2gM_f$. However, the pole at $q^2 = 0$ is spurious and should not show in the physical scattering amplitudes. Thus we see that the pole that was at the threshold in the absence of the minimal coupling moves toward $q^2 = 0$. A further excursion in this direction defines a critical point where the pole appears at $q^2 = 0$ just before disappearing,

$$e_c^2 N = 6M_f^2 g^2 N. \quad (33)$$

The theory is tachyon-free.

4. The anomaly

The careful reader might have already noticed that ignoring the anomaly term at the beginning as an explicit symmetry breaking term might not be plausible since, in principle, we are working with an infinite number of fermion fields. The discrete symmetry $\Psi \rightarrow \gamma_5 \Psi$ will yield the following term when $e \neq 0$,

$$\frac{eN}{2\pi} \frac{\pi}{2} \epsilon_{\mu\nu} F^{\mu\nu}. \quad (34)$$

Here $\pi/2$ is the angle we should use to cast the discrete chiral transformation in S as part of an axial $U(1)$ symmetry. Now, we cannot demand the term above to be finite if we keep $e^2 N$ finite using the large N argument. This difficulty can be remedied, however, by enriching the flavor structure of the theory. If we

have a flavor structure the anomaly term will become

$$\sum_i \frac{e_i N_i}{2\pi} \alpha_i \epsilon_{\mu\nu} F^{\mu\nu}. \quad (35)$$

This sum can be made finite and smaller (even in the case of infinite number of “total” fermion fields) than M_F by a suitable choice of the parameters. Obviously the SSB part will remain unaffected by this and the results of the previous sections will still hold.

5. Conclusion

In this work we have shown that it is possible to achieve Gross–Neveu model from gauge fields only. In the limit where the fermion charge vanishes exactly the correspondance with the Gross–Neveu model is one-to-one. That is there is spontaneous breaking of a discrete chiral symmetry and dynamical mass generation. We argued that the turning on of electric charge and consequently the explicit breaking of the mentioned symmetry by the anomaly can be controlled by extending the flavor structure of the theory. Then, it is possible to treat the anomaly term as a perturbing explicit symmetry breaking term and we see that the fermion–antifermion bound state mass is lowered by an amount proportional to the ratio of the two scales of symmetry breaking.

It would be interesting to test the conclusions about the model proposed in this brief report on the lattice. A joint effort on this is in progress [5].

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